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THE GRAPHS OF EXPONENTIAL SUMS

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§1. *Introduction.* In [3], D. H. Lehmer has analysed the incomplete Gaussian sum

$$G_q(N) = \sum_{j=0}^{N} e(j^2/q),$$

where N and q are positive integers with $N < q$ and $e(x)$ is an abbreviation for $e^{2\pi i x}$. The crucial observation is that, for almost all values of N, $G_q(N)$ is in the vicinity of the point $\frac{1}{4}(1+i)q^{1/2}$. This leads to sharp estimates of the shape $G_q(N) = O(q^{1/2})$.

The behaviour of the Gaussian sum is typical of the exponential sum

$$S(N) = \sum_{j=0}^{N} e(f(j)).$$

If N is restricted to an interval on which $f''(x)$ is small, then $S(N)$ is in the vicinity of certain well-defined "condensation points" for almost all N. As $f''(x)$ becomes larger, the behaviour of $S(N)$ becomes more and more random. The exponential sums arising from $f(x) = (\log x)^k$ illustrate these assertions very clearly. A rather whimsical account featuring these particular sums may be found in [4]. On the other hand, the Weyl sums with $f(x) = \alpha x^k$ and $k > 2$ are not amenable to this type of approach.

The aim of this paper is to give a detailed description of the exponential sum

$$S(N,t) = \sum_{j_n=0}^{N} e(tn^{1/2}),$$

where N is a positive integer and t is a positive real number. This particular sum is of interest because it effectively controls the size of the error term in the classical circle and divisor problems. Thus, in his treatment of the circle problem, Landau [2], Satz 551, obtains the estimate

$$S(N,t) = O(N^{\varepsilon}(N^{71/80}t^{1/40} + N^{79/80}t^{-1/40})),$$

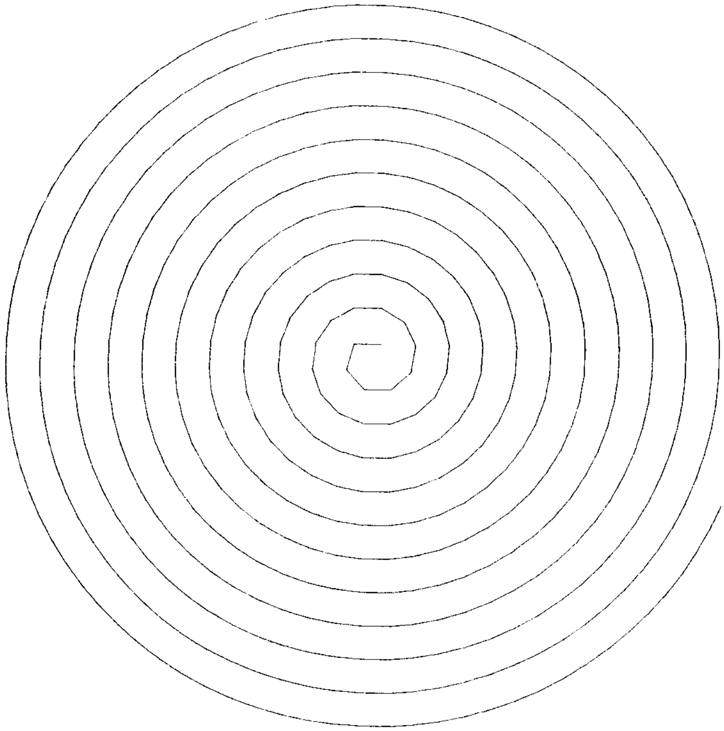

Figure 1. $t = 0{\cdot}5$, $0 \leqslant N \leqslant 500$.

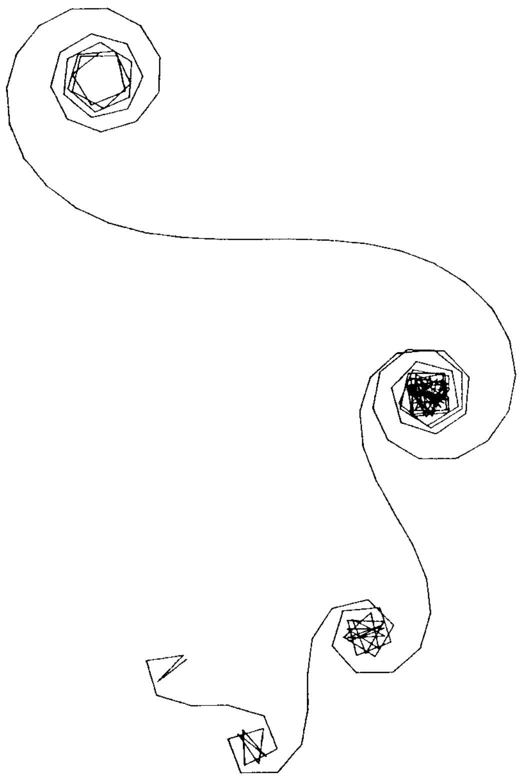

Figure 2. $t = 50$, $20 \leqslant N \leqslant 200$.

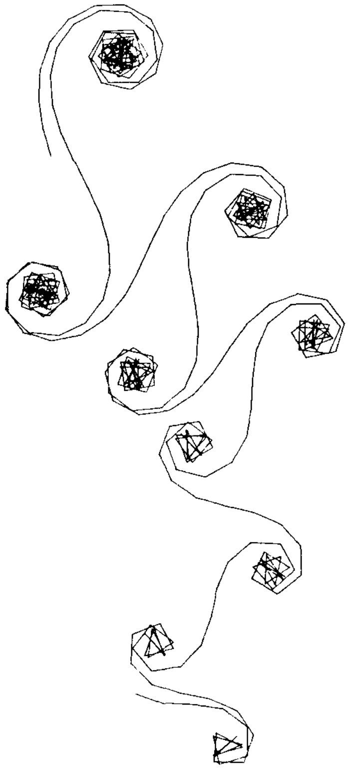

Figure 3. $t = 500$, $155 \leqslant N \leqslant 515$.

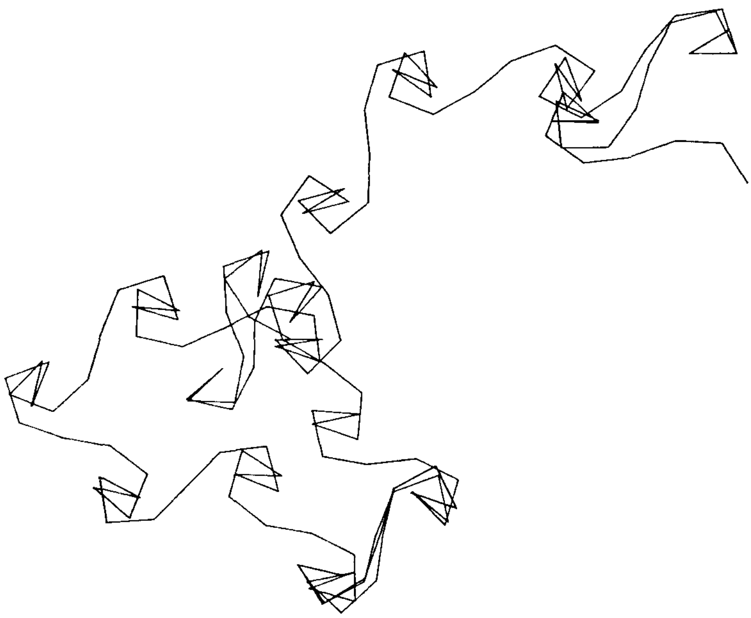

Figure 4. $t = 5000$, $420 \leqslant N \leqslant 580$.

valid for any $\varepsilon > 0$. The description of $S(N, t)$ in terms of "condensation points" works well for $N \geqslant t^{2/3}$, as illustrated in Figures 1 and 2. These pictures are explained by Theorems 1, 2 and 3. For smaller values of $\log N/\log t$, as in Figures 3 and 4, the "condensation points" no longer dominate the picture. However, by using the method to analyse certain auxiliary sums, it is possible to improve Landau's estimate for $N \geqslant t^{34/73}$. This is done in Theorem 4. Further analysis along these lines can be used to recapture Landau's result, giving a non-trivial estimate for $S(N, t)$ for $N \geqslant t^{2/9}$, but it becomes more and more difficult to obtain such estimates as $\log N/\log t$ decreases. It seems that quite different methods are required to deal with the chaotic behaviour of the early terms in the sum.

I thank Scott Slack-Smith for locating the interesting pictures which inspired the work reported here.

§2. *The arcs.* The smooth parts of the graphs will be explained by using the Euler–Maclaurin summation formula to approximate the sum.

LEMMA 1. *Let M and N be integers and suppose that the function $f(x)$ has $2v+1$ continuous derivatives on the interval $M \leqslant x \leqslant N$. Then*

$$\sum_{n=M}^{N} f(n) = \int_{M}^{N} f(x)dx + \tfrac{1}{2}\{f(M)+f(N)\} + \left[\sum_{j=1}^{v} \frac{B_{2j}}{(2j)!} f^{(2j-1)}(x)\right]_{M}^{N} + R_v,$$

where

$$R_v = \frac{1}{(2v+1)!}\int_{M}^{N} B_{2v+1}(x-[x])f^{(2v+1)}(x)dx$$

and the B_j and $B_j(x)$ are the usual Bernoulli numbers and functions respectively.

See, for example, [5], Chapter 2, Section 7.

Eventually, the graph of $S(N, t)$ settles down to a smooth expanding spiral described by $S(N, t) \sim (N^{1/2}/\pi it)e(tN^{1/2})$. This part of the graph is given more precisely by the following theorem.

THEOREM 1. *Fix ε with $0 < \varepsilon < 1$. Let M and N be integers satisfying $t^{2/(1-\varepsilon)} \leqslant M \leqslant N$. Then*

$$S(N, t) - S(M, t) = \frac{1}{\pi it}\{N^{1/2}e(tN^{1/2}) - M^{1/2}e(tM^{1/2})\}$$

$$+ \left(1 + \frac{1}{2\pi^2 t^2}\right)\{e(tN^{1/2}) - e(tM^{1/2})\} + O(t/M^{1/2}).$$

The implied constant may depend on ε but not on M, N or t.

Proof. Apply Lemma 1 to the function $f(x) = e(tx^{1/2})$, taking ν to be the least integer greater than $1/\varepsilon$. For $1 \leqslant j \leqslant 2\nu+1$ and $M \leqslant x \leqslant N$,

$$f^{(j)}(x) = O\big((t/x^{1/2})^j + t/x^{j-(1/2)}\big) = O(t/M^{1/2}).$$

Consequently, the remainder R_ν in Lemma 1 satisfies

$$R_\nu = O\left(\int_M^\infty \{(t/x^{1/2})^{2\nu+1} + t/x^{2\nu+(1/2)}\}\,dx\right) = O(t/M^{1/2})$$

and

$$S(N,t) - S(M,t) = \sum_{n=M+1}^{N} e(tn^{1/2})$$

$$= \int_M^N e(tx^{1/2})dx + \tfrac{1}{2}\{e(tN^{1/2}) - e(tM^{1/2})\} + O(t/M^{1/2}).$$

The integral here is elementary; indeed

$$\int e(tx^{1/2})dx = \left\{\frac{x^{1/2}}{\pi i t} + \frac{1}{2\pi^2 t^2}\right\} e(tx^{1/2}),$$

and this leads to the estimate in the theorem.

Before reaching the final phase described in Theorem 1, the graph follows a sequence of Cornu spirals. In fact, it is possible to identify a Cornu spiral corresponding to each integer p in the range $1 \leqslant p \leqslant t^{2/3}$. The p-th Cornu spiral has width proportional to $t/p^{3/2}$ and is centred about the term with $n \approx t^2/4p^2$ in the sum $S(N,t)$. This is the substance of the next theorem. For the present purposes, the Cornu spiral is the locus in the complex plane described by the point

$$C(s) = \int_0^s e(-\tfrac{1}{4}u^2)du, \qquad -\infty < s < \infty.$$

See, for example, [1], Chapter 3, Section 4.

THEOREM 2. *Fix ε with $0 < \varepsilon < 1$. Let p be a positive integer less than $(t^2/2)^{1/3}$ and set $\delta = (p^3/t^2)^\varepsilon$. Let M be the nearest integer to $t^2/4p^2$ and let N be an integer satisfying $t^2/4(p+\delta)^2 \leqslant N \leqslant t^2/4(p-\delta)^2$. Then*

$$S(N,t) - S(M,t) = \frac{t}{2p^{3/2}} e(t^2/4p) C\big(2(pN)^{1/2} - t/p^{1/2}\big)$$

$$+ \left(\frac{1}{2} - \frac{1}{2\pi i p}\right)\{e(tN^{1/2}) - e(tM^{1/2})\}$$

$$+ \frac{t}{2p^2}(2pM^{1/2} - t)e(tM^{1/2}) + O(\delta).$$

Proof. Apply Lemma 1 to the function $f(x) = e(tx^{1/2}-px)$, taking v to be the least integer greater than $1/2\varepsilon$. Now $f^{(j)}(x)$ is a linear combination of terms of the shape

$$u'(x)^{a_1}u''(x)^{a_2}u'''(x)^{a_3}\dots e(u(x)),$$

where $u(x) = tx^{1/2}-px$ and $a_1, a_2, a_3, \dots$ are non-negative integers satisfying $a_1+2a_2+3a_3+\dots = j$. Since $u^{(j)}(x) = O(t/x^{j-(1/2)})$ for $j \geqslant 2$, the above term is

$$O(u'(x)^{a_1}t^k/x^{j-a_1-(k/2)}),$$

with $k = a_2+a_3+\dots$. Thus, for $0 \leqslant j \leqslant v$ and x between M and N,

$$f^{(2j+1)}(x) = O((p-t/2x^{1/2})^{2j+1}+t^j/x^{(3j/2)+1}) = O(\delta).$$

Since $N-M = O(\delta t^2/p^3)$, the remainder R_v in Lemma 1 satisfies

$$R_v = O\left(\int_M^N \{\delta^{2v+1}+t^v/x^{(3v/2)+1}\}dx\right) = O(\delta)$$

and

$$S(N,t)-S(M,t) = \sum_{n=M+1}^{N} e(tn^{1/2}-pn)$$

$$= \int_M^N e(tx^{1/2}-px)dx+\tfrac{1}{2}\{e(tN^{1/2})-e(tM^{1/2})\}+O(\delta).$$

With the aid of the substitution $u = 2px^{1/2}-t$, this integral reduces to

$$\frac{t}{2p^2}\,e\left(\frac{t^2}{4p}\right)\int_a^b e\left(-\frac{u^2}{4p}\right)du-\left[\frac{1}{2\pi ip}\,e\left(\frac{t^2-u^2}{4p}\right)\right]_a^b,$$

where $a = 2pM^{1/2}-t$ and $b = 2pN^{1/2}-t$. Here $a = O(p^2/t)$ and, by means of an integration by parts,

$$\int_0^a e(-u^2/4p)du = [ue(-u^2/4p)]_0^a+O(a^3/p),$$

so the lower terminal in the preceding integral can be shifted to 0. Finally,

$$\int_M^N e(tx^{1/2}-px)dx = \frac{t}{2p^{3/2}}\,e\left(\frac{t^2}{4p}\right)C\left(\frac{2pN^{1/2}-t}{p^{1/2}}\right)$$

$$-\frac{1}{2\pi ip}\{e(tN^{1/2})-e(tM^{1/2})\}$$

$$+\frac{t(2pM^{1/2}-t)}{2p^2}\,e(tM^{1/2})+O(p^3/t^2),$$

giving the estimate in the theorem.

§3. *The condensation points.* The mechanism responsible for the condensation points in the graph is the same as that occurring in Lehmer's treatment of incomplete Gaussian sums ([**3**], Theorem 3).

LEMMA 2. *Fix $\delta > 0$. Let G be the graph formed by taking successive vertices at the points $S(n) = \sum_{0 \leqslant j \leqslant n} e(u(j))$ for $n = 0, 1, 2, \ldots, N$. Suppose that $\delta(n) = u(n+1) - u(n)$ is increasing and satisfies $\delta \leqslant \delta(n) \leqslant \frac{1}{2}$ for $1 \leqslant n \leqslant N$. Then G lies inside the circle Γ with radius $\frac{1}{2} + \frac{1}{2}\operatorname{cosec} \pi\delta$ and centre $S(0) + (\frac{1}{2} + \frac{1}{2} i \cot \pi\delta) e(u(1))$.*

Proof. In going from the vertex $S(n)$ to the next vertex $S(n+1)$ of the graph, the new edge makes an angle with respect to the previous edge of $2\pi\delta(n)$. If all the $\delta(n)$ were equal to δ, then the vertices of G would lie on a circle of radius $\frac{1}{2}\operatorname{cosec} \pi\delta$ and with the same centre as Γ. In fact, the successive edges of the graph turn more and more inward until the graph is oscillating across a small circle of diameter close to 1. To formalize this argument, it will be shown, by induction on N, that G lies inside the circle with radius $\frac{1}{2}\operatorname{cosec} \pi\delta(1) + \frac{1}{2}\tan \frac{1}{2}\pi\delta(N-1) - \frac{1}{2}\tan \frac{1}{2}\pi\delta(1)$ and with the same centre as Γ. This is clearly true when $N = 2$, since the three vertices then lie on the required circle. Suppose it is true for the graph formed by the N vertices $S(1), S(2), \ldots, S(N)$, and consider the effect of adding one extra vertex $S(0)$. Let Γ_1 and Γ_2 be the circles determined respectively by the points $S(0), S(1), S(2)$ and $S(1), S(2), S(3)$. Then Γ_2 lies inside Γ_1, except for the arc between $S(1)$ and $S(2)$ which, at its centre, is a distance $\frac{1}{2}\tan \frac{1}{2}\pi\delta(2) - \frac{1}{2}\tan \frac{1}{2}\pi\delta(1)$ outside Γ_1. By the induction hypothesis, the whole graph lies inside the circle with the same centre as Γ_1 and radius $\frac{1}{2}\operatorname{cosec} \pi\delta(1) + \frac{1}{2}\tan \frac{1}{2}\pi\delta(N-1) - \frac{1}{2}\tan \frac{1}{2}\pi\delta(1)$, as required.

It is now easy to locate condensation points at the ends of each of the arcs described in Theorem 2. There is a condensation point corresponding to each integer p in the range $1 \leqslant p \leqslant t^{2/3}$; the p-th of these is centred about the term with $n \approx t^2/4(p-\frac{1}{2})^2$ in the sum $S(N, t)$.

THEOREM 3. *Fix ε with $0 < \varepsilon < 1$. Let p be a positive integer less than $(t^2/2)^{1/3}$ and set $\delta = (p^3/t^2)^{\varepsilon}$. Let M be the nearest integer to $t^2/4(p-\frac{1}{2})^2$ and let N be an integer satisfying $t^2/4(p-\delta)^2 \leqslant N \leqslant t^2/4(p-1+\delta)^2$. Then $S(N, t)$ lies inside a circle with centre $S(M, t)$ and radius $1 + \operatorname{cosec} \pi\delta$.*

Proof. Suppose first that $N \leqslant M$. Apply Lemma 2 to the graph corresponding to the sum

$$S(M, t) - S(N, t) = \sum_{n=N+1}^{M} e(tn^{1/2} - pn).$$

In the notation of the lemma,

$$\delta(n) = t(n+1)^{1/2} - tn^{1/2} - p = \frac{t}{2n^{1/2}} - p - \frac{t}{4n} + \ldots$$

lies between $-\frac{1}{2}$ and $-\delta$. The hypotheses of the lemma are satisfied, so all the vertices of the graph lie inside a circle of radius $\frac{1}{2} + \frac{1}{2}\operatorname{cosec} \pi\delta$. The same holds for the graph obtained from the vertices with $N \geqslant M$. The union of these two circles is contained within the circle specified in the theorem.

§4. *An auxiliary sum.* The exponential sum

$$T(M,N;t,h) = \sum_{n=M+1}^{N} e\big(t(n+h)^{1/2} - tn^{1/2}\big)$$

will be required in the next section. The graph of $T(0,N;t,h)$ also exhibits a pattern of spirals over the range $t^{2/5} \leqslant N \leqslant t^{2/3}$, after which it tends towards a horizontal line.

LEMMA 3. *Fix ε with $0 < \varepsilon < 1$ and suppose both t and h are positive.*

(i) *Let M and N be positive integers satisfying $(th)^{(2/3)(1-\varepsilon)} \leqslant M \leqslant N$. Then*

$$T(M,N;t,h) = \int_M^N e\big(t(x+h)^{1/2} - tx^{1/2}\big)dx + \big[\tfrac{1}{2}e\big(t(x+h)^{1/2} - tx^{1/2}\big)\big]_M^N + O(th/M^{3/2}).$$

(ii) *Let p be a positive integer less that $P = (t^2h^2/2)^{1/5}$ and set $\delta = (p^5/t^2h^2)^{\varepsilon/3}$.*

(a) *Let $\xi = \xi_p \approx (th/4p)^{2/3}$ be the positive root of the equation $x^{-1/2} - (x+h)^{-1/2} = 2p/t$. Let M be the nearest integer to ξ and let N be an integer such that $|N-M| \leqslant \delta(t^2h^2/p^5)^{1/3}$. Then*

$$T(M,N;t,h) = \frac{2}{u''(\xi)^{1/2}} e\big(u(\xi)\big)\bar{C}(b) + \left[\left(\frac{1}{2} - \frac{u'''(\xi)}{6\pi i u''(\xi)^2}\right) e\big(u(x)\big)\right]_M^N$$

$$+ \frac{a}{2u''(\xi)^{1/2}} e\big(u(M)\big) + O(\delta),$$

where $u(x) = t(x+h)^{1/2} - tx^{1/2} + px$ and a and b are defined by $a^2 = 4\big(u(M) - u(\xi)\big)$ and $b^2 = 4\big(u(N) - u(\xi)\big)$.

(b) *Let M and N be positive integers between ξ_p and $(th/4\delta)^{2/3}$, such that the interval $M \leqslant x \leqslant N$ is disjoint from the intervals centred around the numbers ξ_p which have been considered in* (a)*. Then $|T(M,N;t,h)| \leqslant 1 + \operatorname{cosec} \pi\delta$.*

Proof. (i) Apply Lemma 1 to the function $f(x) = e\big(t(x+h)^{1/2} - tx^{1/2}\big)$, taking v to be the least integer greater than $1/3\varepsilon$. For $1 \leqslant j \leqslant 2v+1$ and $M \leqslant x \leqslant N$,

$$f^{(j)}(x) = O\big((th/x^{3/2})^j + th/x^{j+(1/2)}\big) = O(th/M^{3/2}).$$

Thus

$$T(M,N;t,h) = \int_M^N e\big(t(x+h)^{1/2} - tx^{1/2}\big)\,dx + \big[\tfrac{1}{2}e\big(t(x+h)^{1/2} - tx^{1/2}\big)\big]_M^N + O(th/M^{3/2}).$$

The integral here is $(N-M)\{1 + O(t^2h^2/M^{1/2})\}$ as $M \to \infty$.

(ii) (a) Apply Lemma 1 to the function $f(x) = e\big(t(x+h)^{1/2} - tx^{1/2} + px\big)$, taking

v to be the least integer greater than $3/2\varepsilon$. For $0 \leqslant j \leqslant v$ and x between M and N,

$$f^{(2j+1)}(x) = O\left(\left(p - \tfrac{1}{2}t(x^{-1/2} - (x+h)^{-1/2})\right)^{2j+1} + (th)^j/x^{(5j/2)+1}\right) = O(\delta),$$

so

$$T(M, N; t, h) = \int_M^N e\big(t(x+h)^{1/2} - tx^{1/2} + px\big)dx + \tfrac{1}{2}\big[e\big(t(x+h)^{1/2} - tx^{1/2}\big)\big]_M^N + O(\delta).$$

To deal with this integral, set $u(x) = t(x+h)^{1/2} - tx^{1/2} + px$ and expand $u(x)$ about $x = \xi$:

$$u(x) - u(\xi) = \tfrac{1}{2}u''(\xi)(x-\xi)^2\left\{1 + \frac{u'''(\xi)}{3u''(\xi)}(x-\xi) + \dots\right\},$$

where the series in parentheses is dominated by a power series in $(x-\xi)/\xi$. With the change of variable given by $v^2 = 4(u(x) - u(\xi))$, the integral becomes

$$(2u''(\xi))^{-1/2} e(u(\xi)) \int_a^b e(\tfrac{1}{4}v^2)\left\{1 - \frac{2u'''(\xi)}{3(2u''(\xi))^{3/2}}v + \dots\right\} dv,$$

where the series is dominated by a power series in $v/\xi u''(\xi)^{1/2}$. The contribution to the above expression from the terms indicated by the dots in the series is $O(\delta)$ and the contribution from the term in v is

$$-\left[\frac{u'''(\xi)}{6\pi i u''(\xi)^2} e(u(x))\right]_M^N.$$

The remaining term of the series yields

$$\frac{2}{u''(\xi)^{1/2}} e(u(\xi)) \int_a^b e(\tfrac{1}{4}v^2)dv = \frac{2}{u''(\xi)^{1/2}} e(u(\xi))\bar{C}(b) + \frac{a}{2u''(\xi)^{1/2}} e(u(M)) + O(\delta).$$

(ii) (b) Take $u(x)$ as in (a) and set $\delta(n) = u(n+1) - u(n)$. If n falls in one of the intervals considered here, then $\delta(n)$ either lies between $-\frac{1}{2}$ and $-\delta$, or between δ and $\frac{1}{2}$. The required estimate for $T(M, N; t, h)$ therefore follows from Lemma 2.

LEMMA 4. *Let M and N be positive integers with $(th)^{2/5} \leqslant M \leqslant N \leqslant (th)^2$, and set $H = N - M$. Then*

$$T(M, N; t, h) = O\big((th)^{1/2}M^{-5/4}\min\{H, M\}\big) \qquad \text{for } M \leqslant (th)^{2/3}, N \leqslant (th)^{8/9}$$

$$= O\big(N^{3/2}(th)^{-1}\big) \qquad \text{for } M \geqslant (th)^{2/3}.$$

Proof. Suppose first that $(th)^{2/5} \leqslant M \leqslant N \leqslant (th)^{2/3}$. Take $\varepsilon = \frac{1}{2}$ in Lemma 3(ii). The width of the Cornu spiral centred on ξ_p, in the notation of the lemma, together with the condensation points terminating the spiral, is $O((th)^{1/3}p^{-5/6})$. Thus

$$T(M, N; t, h) = O\Bigg(\sum_{th/N^{3/2} \leqslant p \leqslant th/M^{3/2}} (th)^{1/3}p^{-5/6}\Bigg)$$

$$= O((th)^{1/2}M^{-5/4}\min\{H, M\}).$$

If, on the other hand, $(th)^{2/3} \leqslant M \leqslant N \leqslant (th)^2$, then part (b) of Lemma 3 (ii) with $\delta = thN^{-3/2}$ gives $T(M, N; t, h) = O(N^{3/2}(th)^{-1})$. The lemma follows on combining these two estimates.

§5. *Estimates for* $S(N, t)$. The description of the graph of $S(N, t)$ obtained in Theorems 1 to 3 provides estimates for $S(N, t)$ which are very good when N is sufficiently large with respect to t. It is still possible to obtain non-trivial estimates for smaller values on N by using the following lemma and the description of the sum $T(M, N; t, h)$ of the previous section.

LEMMA 5. *Let M and N be integers with $M \leqslant N$ and set $H = N - M$. Then*

$$|S(N, t) - S(M, t)|^2 = H + \sum_{\substack{h=-H \\ h \neq 0}}^{H} T(M', N'; t, h),$$

where $M' = \max\{M, M+h\}$ *and* $N' = \min\{N, N-h\}$.

Proof. The left-hand side of the identity is

$$\sum_{m,n=M+1}^{N} e(tn^{1/2} - tm^{1/2}).$$

Setting $n = m+h$ in this expression gives the right-hand side.

The next theorem gives the aforementioned estimates for $S(N, t)$.

THEOREM 4. *Let N be a positive integer. Then*

$$S(N, t) = O(N^{7/12}t^{1/6}), \qquad \textit{for } t^{2/5} \leqslant N = t/(\log t)^{3/2},$$

$$S(N, t) = O(N^{1/4}t^{1/2}), \qquad \textit{for } t/(\log t)^{3/2} \leqslant N \leqslant t^2,$$

$$S(N, t) = O(N^{1/2}t^{-1} + t), \quad \textit{for } N \geqslant t^2.$$

Proof. First, estimate $S(N, t) - S(M, t)$ where the integers M and N satisfy $t^{2/5} \leqslant M \leqslant N \leqslant t/(\log t)^{3/2}$ and $H = N - M = O(M^{5/6}t^{-1/3})$. By Lemma 5 and Lemma 4,

$$|S(N, t) - S(M, t)|^2 = O\Bigg(H + \sum_{1 \leqslant h \leqslant M^{3/2}t^{-1}} \frac{M^{3/2}}{th} + \sum_{M^{3/2}t^{-1} \leqslant h \leqslant H} \frac{(th)^{1/2}H}{M^{5/4}}\Bigg)$$

$$= O(M^{5/6}t^{-1/3}).$$

Set $N_j = [j^6/t^2]$. The above estimate can be applied on each of the intervals (N_j, N_{j+1}) for $t^{2/5} \leqslant j \leqslant (t/\log t)^{1/2}$. Thus

$$S(N,t) - S([t^{2/5}],t) = O\left(\sum_{1 \leqslant j \leqslant N^{1/6}t^{1/3}} N_j^{5/12} t^{-1/6}\right) = O(N^{7/12}t^{1/6}).$$

This gives the first assertion of the theorem, since

$$S([t^{2/5}],t) = O(t^{2/5}).$$

Next, suppose that the integer N satisfies $t^{2/3} \leqslant N \leqslant t^2$. Take $\varepsilon = \frac{1}{2}$ in Theorems 2 and 3. From these theorems, the width of the Cornu spiral centred on $t^2/4p^2$, together with the condensation points terminating this spiral, is $O(tp^{-3/2})$. Thus $S(N,t) - S([t^{2/3}],t)$ is bounded by the sum of the widths of the spirals attached to the integers p with $tN^{-1/2} \leqslant p \leqslant t^{2/3}$, namely

$$S(N,t) - S([t^{2/3}],t) = O\left(\sum_{tN^{-1/2} \leqslant p \leqslant t^{2/3}} tp^{-3/2}\right) = O(N^{1/4}t^{1/2}).$$

This gives the second assertion of the theorem.

In particular, the result just proved gives $S([t^2],t) = O(t)$. By Theorem 3 with $\varepsilon = \frac{1}{2}$ and $p = 1$, the graph of $S(N,t)$ for $t^2 \leqslant N \leqslant t^4$ lies in a circle of radius t. Again, the graph of $S(N',t)$ for $t^4 \leqslant N' \leqslant N$ lies in a circle of radius $N^{1/2}/t$. This gives the final assertion of the theorem.

References

1. E. Jahnke and F. Emde. *Tables of functions with formulae and curves* (Dover, 1943).
2. E. Landau. *Vorlesungen über Zahlentheorie*, Volume 2 (Dover, 1947).
3. D. H. Lehmer. Incomplete Gauss sums. *Mathematika*, 23 (1976), 125–135.
4. J. H. Loxton. Captain Cook and the Loch Ness monster. *James Cook Mathematical Notes*, 3 (1981), 3060–3064.
5. H. Rademacher. *Topics in analytic number theory* (Springer, 1973).

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10G10: *NUMBER THEORY; Exponential sums; Estimates on exponential sums.*

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